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LAMINAR WAVE FLOW OF A FILM OF A VISCOPLASTIC LIQUID

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We solve the problem of a laminar wave falling down a vertical surface for a thin film of a viscoplastic Shvedov–Bingham liquid.

Films of liquid falling down a vertical surface have a wave nature for flow rates exceeding a certain critical value. According to the available experimental data, the increase in the coefficients of heat and mass transfer, due to the wave formation, can reach 50% and greater. Such a type of flow is rather frequently encountered in various applications, in particular, in processes and apparatus of chemical technology. Flowing media, processable in such technological processes, e.g., viscoplastic liquids, are often rheologically complex.

We consider laminar wave flow of a thin film of a viscoplastic liquid falling down a vertical surface, which satisfies the rheological Shvedov–Bingham law

$$\tau' = \tau_0 + \mu \frac{\partial u'}{\partial y'}, \quad |\tau'| > \tau_0, \qquad \frac{\partial u'}{\partial y'} = 0, \quad |\tau'| \leqslant \tau_0. \tag{1}$$

Here τ_0 is the yield point and μ is the plastic viscosity.

We consider the case of long waves, i.e., waves whose length is great in comparison with the thickness of the film. We introduce the dimensionless variables and parameters

$$lt = \alpha U t', \; lx = \alpha x', \; ly = y', \; Uu = u', \; \alpha U v = v', \; lh = h', \; \rho U^2 p = p', \tag{2}$$

$$\mathrm{Re} = Ul\frac{\rho}{\mu}, \; \mathrm{Fr} = \frac{U^2}{gl}, \; W = lU^2\frac{\rho}{\sigma}, \; S = \frac{\tau_0 l}{\mu U}, \tag{3}$$

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where l and U are the characteristic thickness of the layer and the velocity; ρ, density; g, the acceleration of gravity; σ, the surface tension; α, the ratio of the characteristic scale with respect to y' to the scale with respect to x'.

We substitute Eqs. (1), (2), and (3) in the equations of motion, the equations of continuity, and the boundary conditions. We assume $\alpha^2 \ll 1$, the Reynolds number Re is moderately large $\alpha \mathrm{Re} = O(1)$, and furthermore we let the Froude number Fr and the Weber number W satisfy the conditions $\alpha \mathrm{Fr} = O(1)$ and $\alpha^2 W^{-1} = O(1)$. In the equations obtained we discard terms of order α^2, and we find: the equations of motion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\alpha \mathrm{Re}}\frac{\partial^2 u}{\partial y^2} + \frac{1}{\alpha \mathrm{Fr}}, \tag{4}$$

$$p = -\frac{\alpha^2}{W}\frac{\partial^2 h}{\partial x^2}; \tag{5}$$

the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0; \tag{6}$$

and the boundary conditions

$$u = 0,\ v = 0 \ \text{ for } \ y = 0, \tag{7}$$

$$\tau = 0,\ \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = v \ \text{ for } \ y = h(x,\ t). \tag{8}$$

In the case being considered here, we can also write out such equations for the following approximations with respect to α^2. Then, the system (4)-(8) will be the zeroth approximation in such a process of solution. As can be seen below, this zeroth approximation has, in addition to a stationary solution, a wave solution, for which we can determine the values of all the wave parameters – velocity, amplitude, and flow rate. The following terms of the expansions in α^2 will give contributions to these values. For small α^2 the contributions will naturally be small. The instability mechanism here is connected with the action of surface tension, which, according to (5), produces a finite longitudinal pressure gradient.

The system of equations (4)-(8) is nonlinear; therefore, it is quite difficult to find the solution. We choose a complete system of functions w(y), satisfying the boundary conditions, and we represent the velocity in the form [1]

$$u = \sum_{k=1}^{\infty} b_k(x,\ t)\, w_k(y). \tag{9}$$

The number of terms that should be retained in (9) depends considerably on how well the functions $w_k(y)$ are chosen. In the cases of smooth variation of velocity with respect to y it is quite sufficient to use the first several terms.

The problem (4)-(8) for a laminar waveless flow regime in the layer has the form

$$\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2} + \frac{1}{\mathrm{Fr}} = 0,$$

u = 0 for y = 0, τ = 0 for y = h and its solution is described by the function

$$u = \begin{cases} \dfrac{\rho g h^2 l^2}{\mu U}\left[\left(1 - \dfrac{z}{h}\right)\dfrac{y}{h} - \dfrac{y^2}{2h^2}\right], & 0 \leqslant y \leqslant \delta, \\[2ex] \dfrac{\rho g h^2 l^2}{2\mu U}\left(1 - 2\dfrac{z}{h} + \dfrac{z^2}{h^2}\right), & \delta \leqslant y \leqslant h, \end{cases} \tag{10}$$

where

$$z = \frac{\tau_0}{\rho g l};\ \delta = h - z. \tag{11}$$

We determine the mean velocity

$$\bar{u} = \frac{1}{h}\int_0^{lh} u\,dy = \frac{\rho g h^2 l^2}{3\mu U}\left(1 - \frac{3}{2}z + \frac{z^3}{2}\right). \qquad (12)$$

As the characteristic length l it is convenient to choose the thickness of the layer h'; then the equation of the free surface will be h = 1. For small values of the parameter z, which is characteristic for weakly plastic liquids, in the second of Eqs. (10) we can neglect the term z^2/h^2 and, all the more, we can neglect the term $z^3/2$ in Eq. (12). Then with acceptable accuracy, the solution (10) can be approximated by the equation

$$u = \frac{\rho g h^2 l^2}{\mu U}\left[\left(1 - \frac{z}{h}\right)\frac{y}{h} - \frac{y^2}{2h^2}\right] = \frac{\rho g h^2 l^2}{\mu U}\left(\frac{y}{h} - \frac{y^2}{2h^2}\right) - Sy, \qquad (13)$$

and Eq. (12) — by the equation

$$\bar{u} = \frac{\rho g h^2 l^2}{3\mu U}\left(1 - \frac{3}{2}\frac{z}{h}\right). \qquad (14)$$

From Eqs. (13) and (14) we find

$$u = \frac{3\bar{u}}{1 - \frac{3z}{2h}}\left(\frac{y}{h} - \frac{y^2}{2h^2}\right) - Sy. \qquad (15)$$

Proceeding from (15) and following (9), we represent the velocity in problem (4) and (8) in the form

$$u = \frac{3\bar{u}(x,\ t)}{1 - \frac{3}{2}\frac{z}{h(x,\ t)}}\left[\frac{y}{h(x,\ t)} - \frac{y^2}{2h^2(x,\ t)}\right] - Sy, \qquad (16)$$

satisfying the boundary conditions (7), (8) and coinciding with the exact solution for the waveless flow regime. We introduce the independent variable $x_1 = x - ct$, and the dimensionless flow rate $q = \int_0^h u\,dy$. Integrating Eq. (4) with account of (5) and Eq. (6) over y from 0 to h, we obtain a system of equations for q(x, t) and h(x, t)

$$h^2\frac{\partial q}{\partial t} - \left(ch - \frac{12}{5}q + \frac{S}{20}h^2\right)h\frac{\partial q}{\partial x} - \left(\frac{6}{5}q^2 + \frac{S}{20}qh^2 - \frac{S^2}{40}h^4\right)\frac{\partial h}{\partial x} - Gh^3\frac{\partial^3 h}{\partial x^3} - Hh^3 + Eq + \frac{S}{2}Eh^2 = 0, \qquad (17)$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(q - ch) = 0. \qquad (18)$$

Here we have introduced the notation (the index for x_1 has been dropped)

$$G = \alpha^2 W;\ H = (\alpha\,\mathrm{Fr})^{-1};\ E = 3(\alpha\,\mathrm{Re})^{-1}. \qquad (19)$$

The quantity c is connected with the phase velocity of the waves $U_\varphi = cU$. The quantities l and U are the scales of the length and of the velocity, which in the given case are expressed proportionally. For l we take the thickness of the layer averaged over x; then the mean value of h will always be equal to unity. For U we take the velocity averaged over y in the cross section in which the thickness of the layer equals l; then the mean dimensionless flow rate $q_0 = 1$, and the dimensional flow rate $Q = Ul$.

We consider a regime of steady-state traveling waves. Assuming $\partial/\partial t = 0$ in Eqs. (17) and (18) we find $q = ch + q_0 - c$ or, taking into account $q_0 = 1$,

$$q = ch - c + 1. \qquad (20)$$

Let

$$h = 1 + \varphi, \qquad (21)$$

where $\varphi(x)$ is a pure periodic function. Then from (17), using (20) and (21), we obtain

$$G(1+\varphi)\frac{d^3\varphi}{dx^3}+\left\{K-\varphi\left[B+\left(\frac{c^2}{5}-\frac{S}{10}-\frac{cS}{4}+\frac{3S^2}{20}\right)\varphi+\right.\right.$$

$$\left.\left.+\frac{S}{10}(S-c)\varphi^2+\frac{S^2}{40}\varphi^3\right]\right\}\frac{d\varphi}{dx}+\left(3H-\frac{S}{2}E+H\varphi\right)\varphi^2+(3H-cE-SE)\varphi+\left(H-E-\frac{S}{2}E\right)=0. \quad (22)$$

Here

$$K=c^2-\frac{12}{5}c+\frac{6}{5}+\frac{S}{20}\left(c+1-\frac{S}{2}\right),\ B=\frac{2}{5}c^2-\frac{S}{10}\left(\frac{c}{2}+1-S\right). \quad (23)$$

We note that

$$S=z\frac{\mathrm{Re}}{\mathrm{Fr}}=3z\frac{H}{E}. \quad (24)$$

For construction of a periodic solution of Eq. (22) we use an expansion in harmonics

$$\varphi=\beta\sin x+\beta^2(\varphi_{20}\sin 2x+\varphi_{21}\cos 2x)+\cdots \quad (25)$$

After substitution of (25) into Eq. (22) we expand it in a Fourier series. Setting equal to zero the expressions for sin kx and cos kx, we obtain relations for determining the expansion coefficients [1].

In the first, linear approximation, i.e., for waves with infinitesimally small amplitude, we obtain

$$H-E-\frac{S}{2}E=0,\ 3H-cE-SE=0,\ c^2-\frac{12}{5}c+\frac{6}{5}+\frac{S}{20}\left(c+1-\frac{S}{2}\right)=G. \quad (26)$$

Hence, with account of (24) it follows that

$$U=\frac{\rho g l^2}{3\mu}\left(1-\frac{3}{2}z\right), \quad (27)$$

$$c=3(1-z)\Big/\left(1-\frac{3}{2}z\right),\ U_\varphi=cU=gl^2\frac{\rho}{\mu}(1-z), \quad (28)$$

$$\alpha^2=\frac{g^2\rho^3l^5}{3\sigma\mu^2}(1-z), \quad (29)$$

$$l^3=\frac{Q\mu}{\rho g}3\Big/\left(1-\frac{3}{2}z\right). \quad (30)$$

Equations (27) and (30) denote that in the first approximation the thickness of the layer will be close to that which holds for the regime of laminar waveless flow (14). Equations (28) determine the phase velocity of the waves, and Eq. (29) determines the wave number.

In the second approximation we have

$$H-E-\frac{S}{2}E+\frac{\beta^2}{2}\left(3H-\frac{S}{2}E\right)=0,\ 3H-cE-SE=0,$$

$$c^2-\frac{12}{5}c+\frac{6}{5}+\frac{S}{20}\left(c+1-\frac{S}{2}\right)=G, \quad (31)$$

$$\varphi_{20}=-\frac{1}{12G}\left(3H-\frac{S}{2}E\right),$$

$$\varphi_{21}=4+\frac{1}{12G}\left[\frac{2}{5}c^2-\frac{S}{10}(2c+1-S)\right].$$

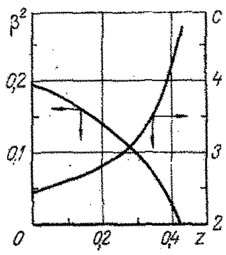

Fig. 1

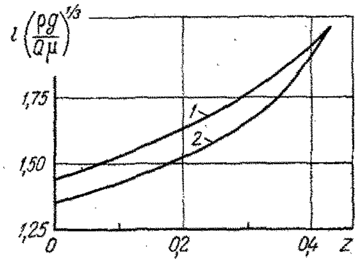

Fig. 2

Fig. 1. Dependence of amplitude of first harmonic β and dimensionless phase velocity c on the platiscity parameter z.

Fig. 2. Effect of plasticity parameter z on mean thickness of film l [(1) linear waves; 2) nonlinear waves].

From system (31), using Eqs. (19) and (24), we obtain

$$1+\frac{3}{2}\beta^2=3\frac{\mathrm{Fr}}{\mathrm{Re}}+\frac{3}{2}z, \tag{32}$$

$$\frac{\mathrm{Re}}{\mathrm{Fr}}=\frac{c}{1-z}, \tag{33}$$

$$c^2-\frac{12}{5}c+\frac{6}{5}+\frac{z}{20}\frac{\mathrm{Re}}{\mathrm{Fr}}\left(c+1-\frac{z}{2}\frac{\mathrm{Re}}{\mathrm{Fr}}\right)=\frac{\alpha^2}{W}, \tag{34}$$

$$\varphi_{20}=-\frac{W}{4\alpha^3\mathrm{Fr}}\left(1-\frac{3}{2}z\right), \tag{35}$$

$$\varphi_{21}=4+\frac{W}{12\alpha^2}\left[\frac{2}{5}c^2-\frac{z}{10}\frac{\mathrm{Re}}{\mathrm{Fr}}\left(2c+1-z\frac{\mathrm{Re}}{\mathrm{Fr}}\right)\right]. \tag{36}$$

The system of equations (32)-(36) contains seven unknowns Re, Fr, c, β^2, α, φ_{20}, φ_{21}. In order to solve the problem, we may assume that two of them are given and the others must be determined from the process of solution. It is convenient to give the parameter c and the Reynolds number, connected with the flow rate $Q = \mu/\rho\mathrm{Re}$. Then, according to Eqs. (32)-(36), we determine the expansion coefficients β, φ_{20}, φ_{21}, the mean thickness l, and the wave number α. The second term on the left side of (32) is connected with the formation of the waves and reflects its effect on the ratio between the flow rate and the mean thickness of the film. For waves of infinitesimally small amplitude, Eq. (32) is transformed into Eq. (27). The mean thickness of the layer is determined from (33)

$$l^3=\frac{Q\mu}{\rho g}\frac{c}{1-z}. \tag{37}$$

From (34) it follows that the real solutions of the problem being considered exist for $\alpha^2 \geq 0$. Substituting $\alpha^2 = 0$ in (34), we obtain c = 1.69. For the linear waves $c = 3(1-z)/(1-{}^3/_2 z)$. Thus, the nonlinear waves in the layer with a free surface propagate with velocity c less than $3(1-z)/(1-{}^3/_2 z)$, but more than 1.69.

Thus, theoretically, for the given flow rate Q there exists an infinite set of wave regimes, which are distinguished by wavelengths and it is impossible to predict beforehand which of them are realized in the experiment. For selection of the primary flow regime we use the hypotheses of Kapitsa [2] about the minimum of viscous dissipation.

The energy being dissipated during flow in the thin flow is equal to

$$-\frac{dE_b}{dt'}=\int_0^{h'}\tau'\frac{\partial u'}{\partial y'}dy'=\frac{U^2\mu}{l}\left(\frac{3q^2}{h^3}+\frac{3}{2}S\frac{q}{h}\right). \tag{38}$$

Averaging the last expression over the wavelength and equating the mean work of the force of gravity to the unit of length $E_n = \rho g Q$, we obtain

$$l^3\left(1-\frac{3}{2}z\Phi\right)=\frac{3\mu Q}{\rho g}F, \tag{39}$$

where

$$F=\frac{1}{2\pi}\int_0^{2\pi}\frac{(1+c\varphi)^2}{(1+\varphi)^3}dx;\ \Phi=\frac{1}{2\pi}\int_0^{2\pi}\frac{1+c\varphi}{1+\varphi}dx. \tag{40}$$

We substitute (25) into (40) and we limit ourselves to the first harmonic; then

$$F=\frac{1}{2}(1-\beta^2)^{-\frac{5}{2}}\{2+\beta^2[1-6c+c^2(1+2\beta^2)]\}, \tag{41}$$

$$\Phi=c-(c-1)(1-\beta^2)^{-\frac{1}{2}}. \tag{42}$$

From an analysis of (39), (41), and (42) it follows that if F takes on the minimum possible value

$$\frac{\partial F}{\partial\beta^2}=0,\ \frac{\partial F}{\partial c}=0 \tag{43}$$

(for given flow rate Q), then the balance of the energy being dissipated and the work of the force of gravity will be satisfied for minimum thickness of the flowing film. The minimum mean thickness of the layer corresponds to the minimum potential energy of the film in a gravitational field and the most stable flow regime.

Then, following Kapitsa [2], we obtain that the condition F = min should correspond to the requirement

$$\frac{c}{1-z}=\frac{3F}{1-\frac{3}{2}z\Phi}, \tag{44}$$

obtained from (37) and (39). Combining (43) and (44), we find the equations

$$\frac{\partial F}{\partial\beta^2}=0,\ \frac{c}{1-z}=\frac{3F}{1-\frac{3}{2}z\Phi}, \tag{45}$$

which determine the sought values of F and Φ, and also c and β. The results of the calculations are represented in Fig. 1. The increase in the yield point for the same flow rate of liquid leads to an increase in the velocity c and a decrease in the amplitude of the nonlinear waves. For a certain value of the plasticity parameter $z = \tau_0/\rho g l = 0.42$, the wave regime of flow in the film is transformed into a waveless flow. Hence, a reinforcement in the plastic properties inhibits wave formation in the film and under definite conditions the wave regime of flow is transformed into a waveless regime. For $0 \le z \le 0.42$ the thickness of the layer l (Fig. 2) decreases in comparison with the waveless regime of flow. The increase in the plastic properties of the liquid increases the thickness of the layer l.

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